Learning through Problems: A Powerful Approach to Teaching Mathematics

Problem solving has been a major focus of mathematics education over the past two decades. Much has been learned about the problem-solving capabilities of elementary school children and their use of strategies in solving problems. However, problem solving has typically existed apart from core curriculum objectives. In many situations, it is a focus only after specific concepts and skills are taught and only then to provide practice for those concepts and skills.

*Principles and Standards for School Mathematics* (NCTM 2000) describes a relatively new role for problem solving. The first goal of the Problem Solving Standard (see fig. 1) places problem solving at the center of teaching and learning. It states that students should “build new mathematical knowledge through problem solving” (NCTM 2000, p. 52). Thus, problem solving is a vehicle by which students make sense of mathematics and learn content, skills, and strategies.

Learning through problems is very different from the mathematical experience of most adults. Adults often associate problem solving with “word problems.” They disliked these problems, did not understand how to solve them, and found them far removed from their world. *Principles and Standards* articulates an approach to learning that uses well-selected problems to capture students’ interest and challenge their intellects:

A problem-centered approach to teaching mathematics uses interesting and well-selected problems to launch mathematical lessons and engage students. In this way, new ideas, techniques, and mathematical relationships emerge and become the focus of discussion. Good problems can inspire the exploration of important mathematical ideas, nurture persistence, and reinforce the need to understand and use various strategies, mathematical properties, and relationships. (NCTM 2000, p. 182)

This article relates an example of the problem-centered approach and discusses ways that it helps all students learn important mathematical concepts and skills.
Learning through Problems: A Fourth-Grade Example

The following problem grew out of a teacher’s experience with two other teachers who rode together to a meeting. The driver had two sticks of gum that he volunteered to share with the others. The teacher explained his experience to his fourth-grade class and asked his students to figure out how much gum each person got if they shared the gum equally. He wrote the problem on the board:

Three people want to share 2 sticks of gum equally. How much gum will they each get?

This lesson occurred early in the year and was the first one to involve fractions. Although the students had not had recent experience with fractions, the teacher felt that the students’ informal knowledge from their out-of-school and third-grade experiences would enable them to solve the problem.

The class reflected on the gum problem for a short time. The teacher then called the students’ attention to various materials that were available (1-inch tiles, fraction strips, paper and markers, and so on). No hints were given about how to get started or what the students should do. The students settled into small groups throughout the room, with a few students choosing to work independently. Soon, the hum of students working and thinking together could be heard. The teacher observed their work closely, making notes about students’ thinking and various strategies. He supported students when they had questions but avoided telling them what to do or giving leading hints. Sometimes he asked them a question about their work. He was also making plans for the “seminar,” or sharing session, that would follow, reflecting on a productive way to sequence the presentations.

After about twenty-five minutes of working on the problem, the students gathered around the teacher to share their strategies and thinking. The class members listened intently as various groups presented their work. Seminars are not “show and tell” sessions; rather, they are occasions for students to build understanding, learn new strategies, and reflect on the ideas of their classmates. The teacher plays a very important role in facilitating the seminar. He must ensure that the discussions and reflections result in learning, and he must be prepared to teach in this context. He might ask a group to clarify or restate an idea, and he invites other children to add to the presenters’ perspectives. He highlights key concepts, such as the notion of a unit and equal-sized pieces, and gives attention to fraction language and notation. The thinking of students is probed, and questions are posed to extend students’ knowledge. He also assesses which students seem unsure of themselves and which ideas need additional attention in future lessons.

The gum-sharing problem was highly productive in helping the students make sense of fractions in their own ways and using their own language. During the seminar, a variety of strategies were presented, misconceptions were corrected, new ideas emerged, and new mathematical connections were established.

The most common strategy used by the students was to show two pieces of gum cut into thirds (see fig. 2a). One group of students wrote initials on the pieces to show the amount of gum that each person got. Most groups reported the answer as 2/3. One group, however, reported that each person got 2/6 of the gum. This solution led to a discussion centering on what the whole was in the problem. If one stick of gum is the unit, then each person got 2/3 of a stick of gum. However, if one person thinks of the two sticks as the whole, then it made sense to this group of students to state that each person got 2/6 of the gum. The teacher was pleased that this issue arose, and he used it to underscore the importance of identifying the unit when working with fractions.

Another approach shared by a group involved dividing each piece of gum in half (see fig. 2b). Thus, each person got 1/2 of a stick of gum initially. Then the remaining half was divided into three pieces. The group concluded that each person got 1/2 + (1/3 of 1/2). The students had difficulty

NCTM’s Problem Solving Standard

Instructional programs from prekindergarten through grade 12 should enable all students to—

• build new mathematical knowledge through problem solving;
• solve problems that arise in mathematics and in other contexts;
• apply and adopt a variety of appropriate strategies to solve problems;
• monitor and reflect on the process of mathematical problem solving.

Source: Principles and Standards for School Mathematics (NCTM 2000)
seeing that $1/3$ of $1/2$ is $1/6$ of the whole. One reason for this confusion might have been that their model did not show a unit divided into six equal parts.

A third approach shared by another group was to break each piece into sixths, with each person getting $4/6$ of a piece of gum (see fig. 2c). This solution opened the door to exploring whether $2/3$ was the same as $4/6$, an introduction to equivalent fractions. As a result of this discussion, the class looked back at the second strategy and revised that solution to $1/2 + 1/6$.

The children had also explored other ideas during their work time. One pair of girls had written $2/3 = 4/6$ and noted that $2$ doubled is $4$ and that $3$ doubled is $6$. They decided to extend the pattern and check to see whether it held: $2/3 = 4/6 = 8/12 = 16/24 = \ldots = 2048/3072$. Another student wrote $1/3 + 1/3 + 1/3 = 1$ for the first model in figure 2a and $2/3 + 2/3 + 2/3 = 2$ for the second model. This use of symbols to record thinking led to further discussion.

This vignette is an example of how teaching through problems can occur. It also captures the mathematical significance of the approach. Important mathematics was considered, fraction ideas and skills were clarified, and the students built new knowledge on existing knowledge and familiar experiences. The lesson also included ideas that are not commonly treated in initial fraction lessons, such as the fact that $1/6$ is half of $1/3$, that $1/2 = 3/6$ and $1/3 = 2/6$, and that $2/3$ is mathematically equivalent to $1/2 + 1/6$.

The role of the teacher, although different from the traditional one, was crucial in causing the work to be productive. It also required solid mathematical knowledge, appropriate decision making based on ongoing assessment of students’ understanding, and judicious management of the discussion to keep the focus on the main ideas. The teacher’s expertise is important. Rich and varied responses from students in this classroom occur when the teacher understands the mathematics well and knows how to draw out initial thoughts from students toward the “next steps.” Teachers benefit from collaborating and reflecting on ways to structure these rich classroom discussions.

## Contributions to Student Learning

Learning through problems is powerful. As students make sense of mathematics, they develop a deep, connected understanding of content and learn essential skills. The approach also causes them to develop as capable, confident problem solvers. The results from various tests, such as the National Assessment of Educational Progress (Kenney and Silver 1997) and the Third International Mathematics and Science Study (U.S. Department of Education 1997), clearly illustrate that although students perform basic skills, such as subtracting three-digit numbers, at a high level, they perform poorly when using knowledge and skills to solve problems. Problem-centered learning contributes to student learning in several ways.

1. The use of familiar, everyday contexts enables children to connect their informal, out-of-school knowledge with school mathematics. The richness of this knowledge is well documented, and it provides a foundation for learning many mathematical ideas. Kindergarten children, for example, use their informal knowledge and ability to interpret the actions in a problem to solve addition, subtraction, multiplication, and division problems (Carpenter et al. 1993). For example, five- and six-year-old children enjoy solving division problems from *The Doorbell Rang* (Hutchins 1986). They find how many cookies each child will get when 12 cookies are shared among 4 children, then work on sharing a handful of small cookies among the 5 members of their groups. They con-
nect the idea of sharing with their knowledge of addition. They know that each of 4 children will get 3 of the 12 cookies by thinking that $3 + 3 + 3 + 3 = 12$. Second graders can solve a problem about sharing 150 cookies among 6 children by using number relationships. They know that $150 \div 3 = 50$ because $50 + 50 + 50 = 150$; they then split each 50 in half to get 25 cookies for each person. The diagram in figure 3 illustrates this approach.

2. A strong focus on reflecting and communicating becomes obvious from observing children as they work in small groups and from observing their seminar discussions. The small-group work and seminars provide extended opportunities to reflect on their thinking and the thinking of others and to communicate their findings to others. Hiebert and others (1997) discuss the crucial roles of reflection and communication in building understanding. They suggest that “stopping to think carefully about things, to reflect, is almost sure to result in establishing new relationships and checking old ones” and that communication “encourages us to think more deeply about our own ideas in order to describe them more clearly or to explain or justify them” (p. 5). The time devoted to solving problems and sharing strategies in problem-centered classrooms enables all children to reflect on ideas from multiple perspectives and to acquire mature ways of thinking over time.

3. Strategies are learned naturally through solving problems. Children possess a variety of problem-solving and computational strategies that they have learned on their own. When they solve problems, a variety of approaches emerge from them without formal instruction. Graham, a first-grade student, thought about how 83 marbles could be shared equally with the 25 students in his class. He reasoned, “Eighty-three is just a little more than seventy-five, so we only get three. There are four quarters in a dollar. There are three quarters in 75 cents. So we can only get three” (NCTM 2000, p. 121). Primary-grade students learn a variety of appropriate strategies for addition facts, and intermediate-grade students learn similar strategies for multiplication facts. Mature strategies actually cause students to memorize facts more effectively. Their invented or mental strategies lead naturally to efficient and meaningful use of paper-and-pencil algorithms for whole numbers. This ongoing focus on mature ways of thinking and efficient ways of computing contributes to computational fluency (Russell 2000; Trafton and Thiessen 1999).

4. Students have multiple opportunities to learn. Typically, learning is expected to occur in a single lesson or a short sequence of lessons. When teachers regularly embed mathematics in problems and tasks and when the problems promote making connections across areas of mathematics, children have multiple opportunities to encounter important ideas. Measurement, place value, geometry, and division, for example, can be considered throughout the year, thus allowing a wide “window of opportunity” for learning. This strategy is particularly effective when children can learn strategies over time. Some children need extended exposure to a strategy before they are ready to use it regularly.

5. Students develop confidence in themselves as problem solvers and become mathematical risk-takers. In classrooms that stress sense-making, reflection, and communication, children learn to listen to one another, respect the thinking of others, and become confident in their capabilities as mathematics students. NCTM’s Professional Standards for Teaching Mathematics (1991) emphasizes classrooms as communities of learners and states, “Students’ learning of mathematics is enhanced in a learning environment that is built as a community of people collaborating to make sense of mathematical ideas” (p. 58). Not only do students learn more mathematics at a deeper level, they learn to value and trust their own thinking and the thinking of others; this mindset carries over into later learning.

Reflections and Guidelines

Problem-centered learning is an exciting and powerful approach to teaching mathematics, and increasing evidence shows its effectiveness in promoting learning. Much is also being learned about how its effectiveness can be maximized.

First, the problems and tasks that are used are crucial. They need to be mathematically rich and appropriate for the goals of the lesson. For example, 100 Hungry Ants (Pinczes 1999) is an excellent choice of a book to introduce multiplication. It
uses arrays, an important model for multiplication. Thus, the book can be used to launch and highlight these mathematical ideas. Selecting good problems is one of the most important tasks for teachers and can be the focus of staff development sessions.

Second, when using problems as the center of learning, a shift takes place from “doing” the activity to “thinking” about relationships among mathematical ideas and how they connect to help students make sense of these ideas. Activities and manipulatives are still important, but the focus needs to be on the mental activity of the students. A helpful view of activities and manipulatives is that of important tools that help students learn, not as ends in themselves.

Third, attention needs to be given to the ways that skills can be developed and learned as an integrated part of this approach. Because the skill work may not be as visible as in a more traditional approach, attention needs to be given to how often it is embedded in the work. Attention also needs to be given to how computational strategies with small and large numbers are learned over time as students repeatedly use them. Teachers need to be aware that students’ invented approaches lead naturally to efficient written algorithms.

Finally, we need to provide ongoing support for teachers as they learn a powerful, yet complex, teaching strategy. They need time to process new knowledge and connect it with their prior knowledge and experience. They need help in dealing with new aspects of decision making in teaching. When teachers receive the support and guidance that they seek, including time to talk among themselves, they become highly proficient. They also gain the benefit of sharing in the delights of their students’ thinking and work.

References